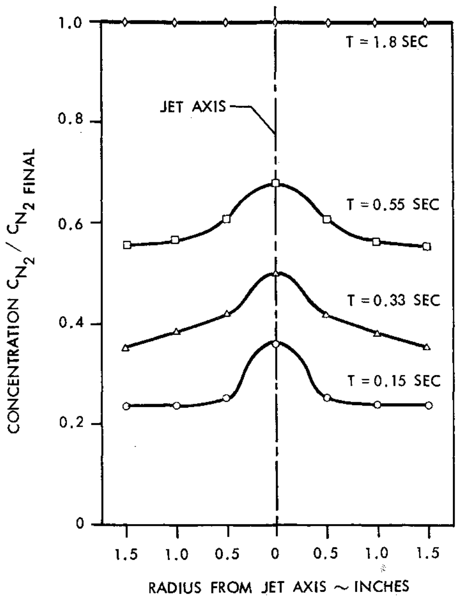

Fig. 3 Nitrogen concentration profiles for direct injection into helium.

equilibrium. These results indicate that by the time all the gas had been transferred to the mixing chamber (approximately 1.8 sec), the nitrogen and helium were uniformly mixed.

Conclusions

The experimental results have verified the linear behavior of the intensity of Raman scattered light as a function of specie concentration. It can be concluded that the technique is capable of identifying and measuring concentrations of individual gaseous species in a mixture during a single 10-nanosec sampling time. By repeating the sampling at the laser rate of 100 pulses per sec, it is possible to monitor a dynamic gas flow situation. For the case of 65 cm^3 of nitrogen at 3000 psi injected into a spherical mixing chamber containing 100 psi of helium, results obtained using the Raman scattering technique illustrate the rapid mixing which is occurring in the vessel. The proven merit of this facility has encouraged the extension of the technique to the study of general gas mixing problems, evaluation of transport properties, and the study of high-pressure, real gas effects.

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Solution of Unsteady Flow of Power Law Fluids

TOMMY Y. W. CHEN* AND DAVID E. WOLLERSHEIM†
University of Missouri, Columbia, Mo.

Nomenclature

B = constant, Eqs. (15) and (21)
C = constant of integration
C_f = coefficient of skin friction
$F(\eta)$ = dimensionless velocity distribution for ISP
$f'(\eta)$ = dimensionless velocity distribution for ISF
K = flow consistency index
N = flow behavior index, or power law fluid index
Re_t = Reynolds number in terms of time
t = time
u = fluid velocity in x-axis direction
U_0 = constant velocity in x-axis direction
x = horizontal coordinate
y = distance from the wall, vertical coordinate
μ = absolute viscosity
ρ = fluid density
ϕ = defined in Eqs. (4) and (6)
τ_{xy} = xy component of shear stress
τ_0 = shear stress at wall
$\tau_0{}^*$ = dimensionless shear stress at wall
δ^* = dimensionless boundary layer thickness
η = similarity parameter defined in Eq. (3a)
ξ = coordinate transformation variable

Subscript

∞ = undisturbed fluid

Introduction

IMPULSIVELY started flow of any power law fluid over a stationary and infinite plate, or an impulsively started plate in any power law fluid moving in its own plane with a constant velocity has been studied analytically.[1–3] Unfortunately, the solutions which can be integrated in closed form were derived only for pseudoplastic and Newtonian fluids (power law fluid index $N \leqq 1$). Recently, Rott[4] presented a solution for an impulsively started plate in a dilatant fluid.

This Note presents the solutions for all power law fluids for both an impulsively started plate and flow cases, including the velocity distribution near the plate, shear stress near the wall, and boundary-layer thickness. The solutions are shown in expressions which can be integrated in closed form. They appear simpler than but are consistent with the results from the previously cited references.

Impulsively Started Plate (ISP)

For an impulsively started plate moving with a constant velocity U_0 in a power law fluid, the velocity gradient is everywhere negative. It is conventional to have a positive shear stress expression for any power law fluid

$$\tau_{xy} = -K|\partial u/\partial y|^{N-1}\,\partial u/\partial y \qquad (1)$$

The equation of motion for the system becomes

$$\rho\,\partial u/\partial t = -\partial\tau_{xy}/\partial y = K\,\partial/\partial y\,(|\partial u/\partial y|^{N-1}\,\partial u/\partial y) \qquad (2)$$

Employing the similarity transformation, let

$$\eta = y\phi(t) \qquad \text{and} \qquad F(\eta) = u/U_0 \qquad (3a, b)$$

where $\phi(t)$ is a function to be found. Substituting Eqs. (3a, b) into Eq. (2), we find

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* Research Assistant, Department of Mechanical and Aerospace Engineering. Associate Member AIAA.
† Associate Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

$$\phi(t) = [1/(N+1)][(K/\rho U_0^{\,1-N})t]^{-1/(N+1)} \tag{4}$$

and the equation of motion becomes

$$(d/d\eta)(|F'|^{N-1}F') + (N+1)^N \eta F' = 0 \tag{5}$$

This is exactly the result derived by Bird.[3] However, it is also possible to let

$$\phi(t) = [2N(N+1)(K/\rho U_0^{\,1-N})t]^{-1/(N+1)} \tag{6}$$

Then the equation of motion becomes

$$(d/d\eta)(|F'|^{N-1}F') + 2N\eta F' = 0 \tag{7}$$

The boundary conditions of the system are $u = U_0$ at $y = 0$, and $u = 0$ at $y = \infty$. The initial condition is $u = 0$ for $t < 0$. The boundary and initial conditions for both Eqs. (5) and (7) are transformed to have

$$F(0) = 1 \quad \text{and} \quad F(\infty) = 0 \tag{8}$$

Equations (4–7) yield the same results as Newtonian flow when $N = 1$ and K is replaced with μ. Here we prefer to use Eqs. (6) and (7) because they have the same similarity variable as in Ref. 1 and Eq. (7) appears simpler than Eq. (5). Since $F'(\eta)$ is everywhere negative, Eq. (7) can be rewritten as

$$(d/d\eta)[(-F')^{N-1}F'] + 2N\eta F' = 0$$

and expanded to become

$$F'' = 2\eta(-F')^{2-N} \tag{9}$$

For $N \neq 1$, Eq. (9) is integrated to yield

$$(-F')^{N-1} = -(N-1)(\eta^2 + C) \tag{10}$$

Note that both sides of Eq. (10) should be positive. Therefore, for $N > 1$ (dilatant fluids), the following condition must be fulfilled:

$$-(\eta^2 + C) \geqq 0 \quad \text{or} \quad C \leqq -\eta^2 \tag{11}$$

Thus we realize C must be always negative and real for all $N > 1$. Let

$$\xi = \eta/|C|^{1/2} \quad \text{where} \quad C \neq 0 \quad \text{and} \quad |C|^{1/2} > 0 \tag{12}$$

Note that C never becomes zero because from Eq. (10), $F'(0) \neq 0$. Substituting Eq. (12) into (10) and rearranging, it becomes

$$-(1/|C|^{1/2})\,dF(\eta)/d\xi = [(N-1)|C|]^{1/(N-1)}(1-\xi^2)^{1/(N-1)} \tag{13}$$

Eq. (13) can be integrated with the boundary condition $F(0) = 1$ to obtain

$$F(\eta) = 1 - B\int_0^{\xi}(1-\xi^2)^{1/(N-1)}\,d\xi \tag{14}$$

or

$$B = [1 - F(\eta)]\Big/\int_0^{\xi}(1-\xi^2)^{1/(N-1)}\,d\xi \tag{14a}$$

where

$$B \equiv |C|^{1/2}[(N-1)|C|]^{1/(N-1)} \tag{15}$$

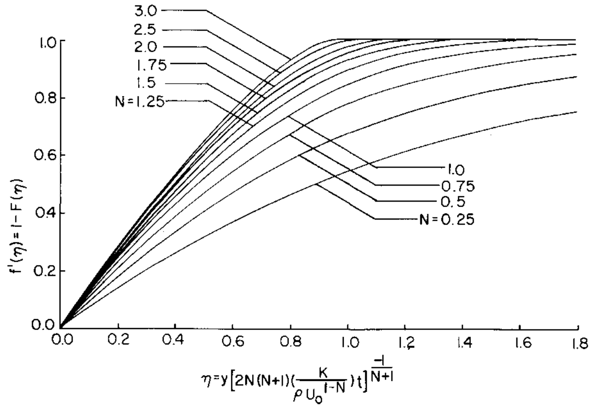

Fig. 1 Velocity distribution $f'(\eta)$ for impulsively started flow and $F(\eta)$ for impulsively started plate in power law fluids.

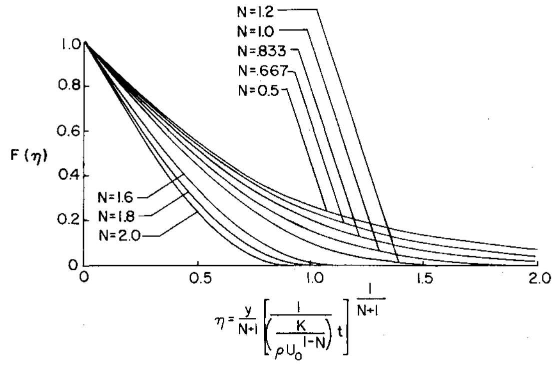

Fig. 2 Velocity distribution for impulsively started plate in power law fluids.

Again note that B is always positive and real because the right-hand side of Eq. (15) is always positive and real.

Rewriting the second boundary condition as

$$\lim_{\eta\to\eta_\infty} F(\eta) = \lim_{\xi\to\xi_\infty} F(|C|^{1/2}\xi) = 0$$

and applying it to Eq. (14a), we find

$$B = [1 - F(\eta_\infty)]\Big/\int_0^{\xi_\infty}(1-\xi^2)^{1/(N-1)}\,d\xi \tag{16}$$

A positive and real value for the right-hand side of Eq. (16) requires that $\xi_\infty = 1$ or

$$\eta_\infty = |C|^{1/2} \tag{17}$$

and

$$B = 1\Big/\int_0^1(1-\xi^2)^{1/(N-1)}\,d\xi \tag{18}$$

Hence Eq. (14) becomes

$$F(\eta) = 1 - \int_0^{\xi}(1-\xi^2)^{1/(N-1)}\,d\xi\Big/\int_0^1(1-\xi^2)^{1/(N-1)}\,d\xi, \quad (0 \leqq \xi \leqq 1) \tag{19}$$

Eq. (19) is the dimensionless velocity distribution for dilatant fluids. By similar reasoning, we find the dimensionless velocity function for pseudoplastic fluids ($N < 1$) as

$$F(\eta) = 1 - \int_0^{\xi}\frac{1}{(1+\xi^2)^{1/(1-N)}}\,d\xi\Big/\int_0^{\infty}\frac{1}{(1+\xi^2)^{1/(1-N)}}\,d\xi, \quad \text{for } N < 1 \tag{20}$$

with

$$B = \frac{C^{1/2}}{[(1-N)C]^{1/(1-N)}} = \left[\int_0^{\infty}\frac{1}{(1+\xi^2)^{1/(1-N)}}\,d\xi\right]^{-1} \tag{21}$$

Table 1 Numerical values of flow parameters and constants

N	B	$C_{ISF} = -C_{ISP}$	C_f	$f''(0) = -F'(0)$	δ^*	η_∞
0.25	0.8959	−1.8079	0.9925	0.6663	8.3765	
0.50[a]	1.2733	−2.1449	0.8145	0.8694	6.4732	
0.75	2.0372	−3.9790	0.6718	1.0213	4.5930	
1.00[b]			0.564	1.128	3.64	
1.25	2.4609	4.1887	0.4823	1.2024	3.1372	2.0466
1.50	1.8750	2.2388	0.4187	1.2531	2.7963	1.4963
1.75	1.6344	1.6115	0.3683	1.2875	2.5699	1.2695
2.00	1.5000	1.3104	0.3276	1.3104	2.4039	1.1447
2.50	1.3529	1.0278	0.2663	1.3345	2.1714	1.0138
3.00	1.2733	0.9003	0.2228	1.3419	2.0133	0.9489

[a] In good agreement with Wells.[1]
[b] Calculated with $F(\eta) = 1 - f'(\eta) = 1 - \text{erf}(\eta)$.

and

$$\xi = \eta / C^{1/2} \tag{22}$$

where C is positive and real and $C^{1/2} > 0$.

It is now possible to derive a skin-friction coefficient $C_f(N)$ as defined by Ref. 5 for Newtonian flow

$$C_f(N) = \tau_0{}^* Re_t{}^{1/(N+1)} = \frac{[-F'(0)]^N}{[2N(N+1)]^{N/(N+1)}} \tag{23}$$

in which $\tau_0{}^* \equiv \tau_0/\rho U_0{}^2$ is the dimensionless shear stress at the wall. Re_t is the Reynolds number in terms of time which can be expressed in a form equivalent to the conventional expression for non-Newtonian flow if we replace t by L/U_0 where L is some characteristic length, such that

$$Re_t = \rho U_0{}^2 t^N/K = \rho U_0{}^{2-N} L^N/K$$

Note that the velocity gradient at the wall $F'(0)$ in Eq. (23) is always negative and can be obtained from Eq. (10) such that

$$[-F'(0)] = [(1-N)C]^{1/(N-1)}, \quad \text{where} \quad C = C(N) \tag{24}$$

The dimensionless boundary-layer thickness defined at $u/U_0 = 0.01$ can now be obtained from Eqs. (3a) and (6) as

$$\delta^* \equiv (y/U_0 t) Re_t{}^{1/(N+1)} = [2N(N+1)]^{1/(N+1)} \eta|_{F(\eta)=0.01} \tag{25}$$

Impulsively Started Flow (ISF)

With a technique similar to that developed for the ISP case, an impulsively started power law fluid flow with a constant velocity U_0 over a stationary plate has the following velocity distribution function

$$f'(\eta) = \int_0^\xi (1-\xi^2)^{1/(N-1)} d\xi \bigg/ \int_0^1 (1-\xi^2)^{1/(N-1)} d\xi = 1 - F(\eta) \quad \text{for} \quad N > 1 \tag{26}$$

and

$$f'(\eta) = \int_0^\xi \frac{1}{(1+\xi^2)^{1/(1-N)}} d\xi \bigg/ \int_0^\infty \frac{1}{(1+\xi^2)^{1/(1-N)}} d\xi = 1 - F(\eta) \quad \text{for} \quad N < 1 \tag{27}$$

where $f'(\eta) = u/U_0$, $\xi = \eta/|C|^{1/2}$, and $|C|$ is the absolute value of C which can be obtained from Eqs. (15) and (21) for $N \gtrless 1$. The skin-friction coefficient can be obtained by replacing $-F'(0)$ with $f''(0)$ in Eq. (23) and the dimensionless boundary-layer thickness is defined at $u/U_0 = 0.99$ for η in Eq. (25).

Discussion

The dimensionless velocity distributions for various power law fluids are calculated and plotted in Fig. 1. The velocity increases for ISF or decreases for ISP and decays much faster when the power law fluid index is higher. The important flow parameters such as skin-friction coefficient C_f, velocity gradient at the wall $f''(0)$ for ISF and $F'(0)$ for ISP, dimensionless boundary-layer thickness δ^*, finite boundary-layer thickness η_∞ and constants B and C are tabulated in Table 1 for various values of N. Figure 1 and Table 1 utilize the similarity transformation of Eq. (6). The results for pseudoplastic fluids are in agreement with those of Ref. 1. A solution of Eq. (5), which utilizes the similarity transformation of Eq. (4), has been obtained and is plotted for reference in Fig. 2.

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Optimization of Simple Structures with Higher Mode Frequency Constraints

Terrence A. Weisshaar*
University of Maryland, College Park, Md.

Nomenclature

$t(x)$ = $T(x)/T_{\text{ref}}$, nondimensional thickness
$t_m(x)$ = nondimensional thickness associated with the mth frequency
x = nondimensional coordinate
α_m = $\delta[(2m-1)/2]^2\pi^2$
β_m = $(1-\delta_1)[(2m-1)/2]^2\pi^2$
δ_1 = ratio between reference structural weight and total reference structure weight, a constant
$\theta(x)$ = cylinder cross section angle of rotation
$(\ )'$ = $d(\ )/dx$

Introduction

THIS Note presents some interesting results of a study of least weight optimization of simple structures with a single natural frequency constraint. This frequency constraint may be any one of the natural frequencies of the structure. Variational techniques are used to derive the necessary equations. Numerical solution methods have been used, where necessary, to find solutions to the nonlinear, two-point boundary value problems which result.

Discussion

Variational calculus provides a convenient and useful tool for the study of simple structural optimization problems. Although this method of analysis is somewhat restricted, the results of such analyses often provide insight into more sophisticated problems. This approach to the search for a least weight structure whose fundamental frequency is fixed at a certain value has been the subject of several articles.[1–4] Various methods have been employed to achieve solutions to this class of problems. In many cases, although the governing equations are easily derived, an analytic solution cannot be found. However, several numerical techniques have been presented which overcome this difficulty. The free torsional vibration of a thin-wall cylinder of length L with variable wall thickness $T(x)$ is discussed in detail by Armand.[5] The necessary conditions for a least weight cylinder with a fixed fundamental torsional frequency are also derived in Ref. 5. A simple extension of Armand's work shows that free vibration in the mth mode can be described by the nondimensional differential equations (1a–c). A nonstructural mass moment of inertia has been added to ensure that the problem is properly posed.

$$\theta_m'(x) = s_m/t \qquad 0 < x < 1 \tag{1a}$$

$$s_m'(x) = -(\alpha_m t + \beta_m)\theta_m(x) \tag{1b}$$

and

$$\theta_m(0) = s_m(1) = 0 \tag{1c}$$

Only the nondimensional wall thickness $t(x)$ is free to be varied. An objective function is defined as

$$J = \int_0^1 t(x)\,dx \tag{2}$$

The problem is to find a function $t(x)$ which yields a stationary value of J and satisfies equations (1a–c). A straightforward extension of this problem is that of fixing several frequencies simultaneously.[6]

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* Assistant Professor, Aerospace Engineering. Associate Member AIAA.